Triple delooping for multiplicative hyperoperads

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Polynomial monads

Let $\mathcal{E}$ be a locally cartesian closed category. A *polynomial* over $\mathcal{E}$ [2] is a diagram in $\mathcal{E}$ of shape

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J. \quad (1)$$

We call *extension* of this polynomial the composite

$$C/I \xrightarrow{\Delta_s} C/E \xrightarrow{\Pi_p} C/B \xrightarrow{\Sigma_t} C/J,$$

where Δ_s is the *base change* along s, Π_p is the *dependent product* along p and Σ_t is the *dependent sum* along t. A *polynomial functor* [2] is any functor isomorphic to the extension of a polynomial. A *polynomial monad* [2] is a monad whose underlying functor is polynomial and whose unit and multiplication are cartesian natural transformations.

Baez-Dolan plus construction

We now focus to the case where $\mathcal{E} = \mathrm{Set}$ is the category of sets. Let P be a polynomial monad, whose underlying functor is the extension of the polynomial 1, with $I = J$. We can construct a new polynomial monad P^+, called *Baez-Dolan plus construction* [1, 3], whose underlying functor is the extension of

$$B \xleftarrow{s^+} tr(P)^* \xrightarrow{p^+} tr(P) \xrightarrow{t^+} B,$$

where $tr(P)$ is the set of P-trees, that is trees whose edges are decorated in I and vertices are decorated in B in a coherent way [3], and $tr(P)^*$ is the set of P-trees with one marked vertex. The map s^+ returns the element decorating the marked vertex, the map p^+ forgets the marking and the map t^+ is given by contracting all the inner edges of the tree and composing the elements of B decorating its vertices accordingly, using the monad structure on P. The multiplication of the monad P^+ is given by insertion of a tree inside a vertex.

Iterating the Baez-Dolan construction from the identity monad on Set, one gets consecutively the *free monoid monad* and the monad for *non-symmetric operads*. Iterating yet again gives a polynomial monad whose algebras we call *hyperoperads*.

Presentation of our result

Any planar rooted tree T generates a non-symmetric operad $\Omega_p(T)$, whose set of colours is the set of edges of T and the operations are generated by the vertices of the tree [4]. The *category of planar rooted trees* Ω_p is defined as the full subcategory of the category of non-symmetric operads whose objects are $\Omega_p(T)$ for any tree T.

For a planar rooted tree T, we write $\alpha(T) = l(T) + 1$, where $l(T)$ is the set of leaves of T. This obviously extends to a functor $\alpha : \Omega_p \to \mathrm{Set}$. We write Ω_p^* for the category of elements of α and $\pi : \Omega_p^* \to \Omega_p$ for the projection.

We call a hyperoperad *multiplicative* when it is equipped with a map from the terminal object. A multiplicative hyperoperad gives rise to a covariant presheaf $\mathcal{O}^\bullet$ over Ω_p. Our main result is the following theorem:

Theorem 1 *Let $\mathcal{O}$ be a multiplicative hyperoperad in the symmetric monoidal category of topological spaces. Assume that $\mathcal{O}(T)$ is contractible for all trees with zero or one vertices. Assume also that* $\mathrm{holim}\, \pi^*(\mathcal{O}^\bullet)$ *is contractible, where π^* is the restriction functor associated to π. Then there is a triple delooping*

$$\Omega^3 \mathrm{Map}_{HOp}(1, u^*(\mathcal{O})) \sim \mathrm{holim}\, \mathcal{O}^\bullet$$

where 1 is the terminal hyperoperad, u^* is the forgetful functor from multiplicative hyperoperads to hyperoperads and $\mathrm{Map}_{HOp}(-,-)$ *is the homotopy mapping space in the category of hyperoperads.*

References

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